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SOR Methods for Coupled Elliptic Partial Differential Equations

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The biharmonic or Navier–Stokes problems in the form of a coupled pair of Dirichlet problems (J. Smith, *SIAM J. Numer. Anal.* **5**, 323 (1968)) are numerically solved by using a two parameter point SOR method. We emphasize the dependance of the convergence domain on the discrete boundary formulae. Optimization of this SOR method is heuristic but can be foreseen with a satisfactory precision. The optimal region is rather large and although using imprecisely optimal parameters, we can greatly improve the classical block SOR method (L. W. Ehrlich, *SIAM J. Numer. Anal.* **8**, 278 (1971); L. W. Ehrlich and M. M. Gupta, *SIAM J. Numer. Anal.* **12**, 773 (1975); M. M. Gupta and R. P. Manohar, *J. Comput. Phys.* **31**, 265 (1979); M. Khalil, Thesis, Université Paul Sabatier, Toulouse, 1983 (unpublished)).

1. INTRODUCTION

Two fundamental problems in the continuum mechanics are set up with a coupled system of two elliptic partial differential equations:

(I) Biharmonic problem,

$$\Delta^2\psi = f_1 \quad \text{reduced to} \quad \Delta\psi = \Omega \qquad \text{on } D \subset \mathbb{R}^n \tag{1}$$

$$\Delta\Omega = f_1 \qquad \text{on } D \subset \mathbb{R}^n \tag{2}$$

with the boundary conditions

$$\frac{\partial\psi}{\partial n} = f, \qquad \psi = g \quad \text{on } \partial D \tag{3}$$

(II) Navier–Stokes problem in $D \subset \mathbb{R}^2$, which reduces to (I) for zero Reynolds number,

$$\Delta\psi = -\Omega \tag{4}$$

$$\Delta\Omega + R\left(\frac{\partial\psi}{\partial y}\frac{\partial\Omega}{\partial x} - \frac{\partial\psi}{\partial x}\frac{\partial\Omega}{\partial y}\right) = \Delta\Omega = f_2 \tag{5}$$

and boundary conditions such as (3).

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The boundary conditions (3) correspond to the first biharmonic problem and to the usual situation for the Navier–Stokes problem which generally prescribes two conditions on the stream function ψ and no condition on the vorticity Ω: the elliptic equations (1) (or (4)) and (2) (or (5)) are then, respectively, over- and underdetermined.

One encounters elliptic systems in other contexts, but a difficulty common to (I) and (II) is the coupling of the partial differential equations by the boundary conditions, this fact being of major importance to the SOR solvers analyzed in this article.

2. Sequence of Dirichlet Problems

D is covered with a regular grid of step $h: D_h$. The discrete problem (I_h) corresponding to (I) (Eqs. (1–3)) is defined on D_h by

(I_h)

$$\Delta_h \psi_h = \Omega_h \qquad \text{on } D_h \tag{6}$$

$$\Delta_h \Omega_h = f_1 \qquad \text{on } D_h \tag{7}$$

$$\Omega_h = M_h \psi_h + k \qquad \text{on } \partial D_h. \tag{8}$$

Δ_h is a finite difference approximation of the Laplacian operator Δ.

Equation (8) results from an approximation of (1) on ∂D_h using the boundary conditions (3) (approximation of the normal derivative); this approximation of the normal derivative must be at least of order $(p-1)$, if p is the order of the approximation used in the elliptic operators (see details in Smith [11], Ehrlich [1], Ehrlich–Gupta [3]).

With the usual fivepoint approximation of Δ, (I_h) leads to the following linear system (k_i depending on f and g upon ∂D_h),

$$A\psi_h = h^2 \Omega_h + k_1 \qquad \text{on } D_h \tag{9}$$

$$A\Omega_h = h^2 f_1 + k_2 \qquad \text{on } D_h \tag{10}$$

$$\Omega_h = \frac{M}{h^2} \psi_h + k_3 \qquad \text{on } \partial D_h, \tag{11}$$

where A is block-tridiagonal and symmetric negative definite, and M is a singular band-matrix. For the first order approximation of the normal derivative, M is a diagonal matrix given by

$$M = \begin{pmatrix} T+I & & & 0 \\ & T & & \\ & & \ddots & \\ & & & T \\ 0 & & & T+I \end{pmatrix} \quad \text{with } T = \begin{pmatrix} 1 & & & 0 \\ & 0 & & \\ & & \ddots & \\ & & & 0 \\ 0 & & & 1 \end{pmatrix}$$

when D is a rectangular region.

In practice, the boundary conditions explicitly written in (11) are used in (10) and (I_h) becomes

(I_h)

$$A\psi_h = h^2\Omega_h + k_1 \qquad \text{on } D_h \tag{12}$$

$$A\Omega_h = -\frac{M\psi_h}{h^2} + k_4 \qquad \text{on } D_h. \tag{13}$$

With second order finite difference approximations of Δ and A (Eq. (5))—five point for Δ, centered (e.g.) for the first derivatives—we can define (II_h) which has the same form as (I_h),

(II_h)

$$A\psi_h = h^2\Omega_h + k_1 \tag{12'}$$

$$B\Omega_h = -\frac{M\psi_h}{h^2} + k_4, \tag{13'}$$

where B is a non-singular block tridiagonal matrix.

An iterative procedure similar to the MSOR method of Young [12] has been introduced by Smith [11], developed in [1, 3] for (I) and in [5, 6] for (II) and is identified as the SQSOR method [2]. The corresponding algorithm is given by

$$A\Omega^{(n+1)} = (1-\omega_1)\, A\Omega^{(n)} - \frac{M\omega_1}{h^2}\,\psi^{(n)} + \omega_1 k_4 \tag{14}$$

$$A\psi^{(n+1)} = (1-\omega_2)\, A\psi^{(n)} + \omega_2(h^2\Omega^{(n+1)} + k_1), \tag{15}$$

and in matrix form

$$\begin{pmatrix} \Omega^{(n+1)} \\ \psi^{(n+1)} \end{pmatrix} = \begin{pmatrix} (1-\omega_1)\, I & -\omega_1 \dfrac{A^{-1}M}{h^2} \\ \omega_2(1-\omega_1)\, h^2 A^{-1} & (1-\omega_2)\, I - \omega_1\omega_2 A^{-2}M \end{pmatrix} \begin{pmatrix} \Omega^{(n)} \\ \psi^{(n)} \end{pmatrix} + K. \tag{16}$$

With an eigenvector $X = (X_\Omega, X_\psi)^t$ partitioned like the blocks in the above iteration matrix $\mathscr{A}_{\omega_1,\omega_2}$, we write the eigenvalue equation

$$\mathscr{A}_{\omega_1,\omega_2} X = \lambda X$$

and, eliminating X_Ω, we obtain

$$[(1-\omega_2-\lambda)(1-\omega_1-\lambda)\, I + 2\omega_1\omega_2\lambda A^{-2}M]\, X_\psi = 0.$$

Thus, if τ is an eigenvalue of $A^{-2}M$, the eigenvalues of the SQSOR method are determined by

$$\lambda^2 - [(1-\omega_2) + (1-\omega_1) - \omega_1\omega_2\tau]\,\lambda + (1-\omega_1)(1-\omega_2) = 0. \tag{17}$$

The main results given in [1, 3], which are important in our context, are the following:

(i) the rate of convergence, $O(\sqrt{h})$, of the SQSOR method (this rate was only $O(h)$ in the first version of this method [11]);

(ii) the optimization of the SQSOR method when

$$\omega_1 = \omega_2 = \frac{2}{1+\sqrt{1+\bar{\tau}}},$$

$\bar{\tau}$ spectral radius of $A^{-2}M$.

We complete these results by the definition of the convergence domain of the SQSOR method. This domain and the algorithmic properties of the method are discussed below.

(A) From (17) we easily obtain the convergence domain of the SQSOR method in the (ω_1, ω_2) plane

$$\Delta = \{(\omega_1, \omega_2): |\lambda(\omega_1, \omega_2)| < 1\}$$

$$\Delta = \{(\omega_1, \omega_2): \omega_1 > 0, \omega_2 > 0, (\omega_1 - 2)(\omega_2 - 2) - \omega_1 \omega_2 \bar{\tau} > 0\}$$

limited by the hyperbola with asymptotes (see Fig. 1)

$$\omega_1 = \omega_2 = \frac{2}{1-\bar{\tau}} = a.$$

We know that, for rectangular D, $\bar{\tau}$ increases when h decreases [1], $\bar{\tau} = O(h^{-1})$; therefore, when h tends to zero, Δ is restricted to the neighborhood of the axes $\omega_1 = 0$, $\omega_2 = 0$. Moreover, the convergence is optimal when [1, 12]

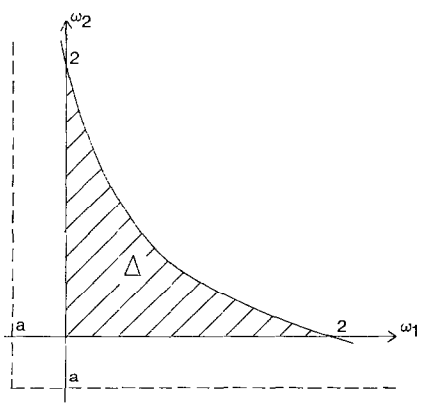

FIG. 1. Convergence domain Δ of the SQSOR method for the biharmonic problem (I_h).

$$\omega_1 = \omega_2 = \frac{2}{1+\sqrt{1+\bar{\tau}}},$$

which, when $\bar{\tau} \gg 1$, is very close to

$$\omega^* = \frac{2}{1+\sqrt{\bar{\tau}}}$$

the upper bound of the convergence domain along the line $\omega_1 = \omega_2$. The choice of the parameters is then very sharp and if we do not know $\bar{\tau}$ precisely, the divergence of the SQSOR method may occur. These observations impose a very cautious utilization of SQSOR solvers.

(**B**) The SQSOR method is a block algorithm: every "outer" iteration (index n in (14–15)) needs two resolutions of linear systems with matrix A (there exist very efficient direct solvers). $\psi^{(n+1)}$, $\Omega^{(n+1)}$ may also be obtained iteratively (a point SOR method seems the most suitable): these iterations will be named "inner." This procedure is not more costly because, when the SQSOR method converges, a few "inner" iterations are sufficient to obtain $\bar{\psi}^{(n+1)}$, $\bar{\Omega}^{(n+1)}$ approximate values of $\psi^{(n+1)}$, $\Omega^{(n+1)}$.

The SQSOR method, as detailed for the biharmonic problem (I_h), works similarly for the Navier–Stokes problem with particular attention to the convective term: centered or upstream approximation according to whether the Reynolds number is moderate or large.

For (II_h) the above remark (**B**) is fundamental: each outer iteration modifies the matrix B and therefore needs a direct or iterative resolution of the linear system (13′) to obtain the successive approximation $\Omega^{(n+1)}$, $\psi^{(n+1)}$.

3. Global Iterative Resolution of the Elliptic System

To avoid the difficulties mentioned above, we consider a global resolution of (I_h), (II_h) with a point iterative method: PSOR. Like point SOR methods, its rate of convergence cannot be better than $O(h)$ and thus compare unfavorably with the SQSOR method. This approach has not actually been developed [2], probably because the matrix formulation, much heavier than (16), does not allow a convenient analysis of the convergence and optimization of the method.

However, the study of the PSOR method may be partly connected with the preceding analysis, and although its optimization remains heuristic, we will observe that the PSOR method is very efficient and easy to use. We write

$$A = D - L - U,$$

where, as usual $D = \mathrm{diag}(A)$ and L and U, respectively, the strictly lower and upper triangular parts of A. So, we define for (I_h) the PSOR algorithm

$$\psi^{(n+1)} = (1-\omega_1)\,\psi^{(n)} + \omega_1 D^{-1}[L\psi^{(n+1)} + U\psi^{(n)} + h^2\Omega^{(n)}] + \omega_1 D^{-1}k_1 \tag{18}$$

$$\Omega^{(n+1)} = (1-\omega_2)\,\Omega^{(n)} + \omega_2 D^{-1}\left[L\Omega^{(n+1)} + U\Omega^{(n)} - \frac{M\psi^{(n+1)}}{h^2}\right] + \omega_2 D^{-1}k_4; \tag{19}$$

hence,

$$\begin{bmatrix}\psi^{(n+1)}\\ \Omega^{(n+1)}\end{bmatrix} = [\mathscr{L}_{\omega_1\omega_2}]\begin{bmatrix}\psi^{(n)}\\ \Omega^{(n)}\end{bmatrix} + K_1.$$

When M is diagonal, i.e., for a first order approximation of the normal derivative on the boundary, the eigenvalue equation $|\mathscr{L}_{\omega_1\omega_2} - \lambda I| = 0$ reduces to [2]

$$|(-A + (\alpha_1+1)\,D)^{-1}(-A + (\alpha_2+1)\,D)^{-1}M + I| = 0, \tag{20}$$

where

$$\alpha_i = \frac{1-\omega_i-\lambda}{\omega_i\sqrt{\lambda}}, \tag{21}$$

a non-linear eigenvalue problem which has the form

$$|A_0\mu^4 + A_1\mu^3 + A_2\mu^2 + A_3\mu + A_4| = 0 \tag{22}$$

with $\mu = \sqrt{\lambda}$ and A_i depending on ω_1, ω_2, and $B_J = A - D$. Equation (20) must be compared with the eigenvalue equation used in Section 2,

$$|A^{-2}M - \tau I| = 0. \tag{23}$$

We cannot deduce λ in terms of τ, but it is possible to obtain a relationship between the spectral radii through discrete scalar products,

$$\begin{aligned}\bar{\tau} &= \sup_{X\neq 0}\left|\frac{\langle MX, X\rangle}{\langle AX, AX\rangle}\right| \\ &= \sup_{X\neq 0}\left|\frac{\langle MX, X\rangle}{\langle(-A+(\alpha_1+1)\,D)\,X, (-A+(\alpha_2+1)\,D)\,X\rangle}\right| \\ &\quad\times\left|\frac{\langle(-A+(\alpha_1+1)\,D)\,X, (-A+(\alpha_2+1)\,D)\,X\rangle}{\langle AX, AX\rangle}\right|,\end{aligned}$$

hence,

$$\begin{aligned}\bar{\tau} &\leqslant \sup_{X\neq 0}\left|\frac{\langle MX, X\rangle}{\langle(-A+(\alpha_1+1)\,D)\,X, (-A+(\alpha_2+1)\,D)\,X\rangle}\right| \\ &\quad\times \sup_{X\neq 0}\left|\frac{\langle(-A+(\alpha_1+1)\,D)\,X, (-A+(\alpha_2+1)\,D)\,X\rangle}{\langle AX, AX\rangle}\right|.\end{aligned}$$

The first factor must be compared with (20) and we reduce the second term because A and $A+(\alpha_i+1)D$ commute, for $D=-4I$ when the grid is regular. Thus,

$$\bar{\tau} \leqslant |1+4(\alpha_1+\alpha_2+2)\,\tilde{\mu}+16(\alpha_1+1)(\alpha_2+1)\,\tilde{\mu}^2|, \tag{24}$$

where $\tilde{\mu}$ is the spectral radius of A^{-1}.

We obtain the most restrictive condition by considering the equality case in (24). Replacing α_1, α_2 in terms of ω_1, ω_2, we proceed by analogy with the analysis in [9], after rather tedious calculations we obtain a sixth order symmetric polynomial equation in $|\sqrt{\lambda}|$. Prescribing $|\sqrt{\lambda}|<1$, we obtain the convergence domain

$$\Delta^* = \{(\omega_1, \omega_2): \omega_1>0,\ \omega_2>0,\ (8\tilde{\mu}-1)(\omega_1-2)(\omega_2-2)-\bar{\tau}\omega_1\omega_2>0\}$$

limited by the axes and the hyperbola with asymptotes (see Fig. 2)

$$\omega_1=\omega_2=\frac{2}{1-\bar{\tau}/(8\tilde{\mu}-1)}=b.$$

Now, we reconsider the remarks **(A)**, **(B)** developed in Section 2 about the SQSOR method

(A*) For two-dimensional problems, $\tilde{\mu}=O(h^{-2})$, $\bar{\tau}=O(h^{-1})$ [1, 3], therefore, when h tends to zero, Δ^* becomes the square $[0,2]\times[0,2]$: the behavior of the PSOR method is asymptotically favourable. This is not surprising, for, when $h\to 0$, the coupling (matrix M) of both Dirichlet problems (12) and (13) has a negligible effect and we know that the PSOR method converges in this case when ω_i belongs to $]0,2[$.

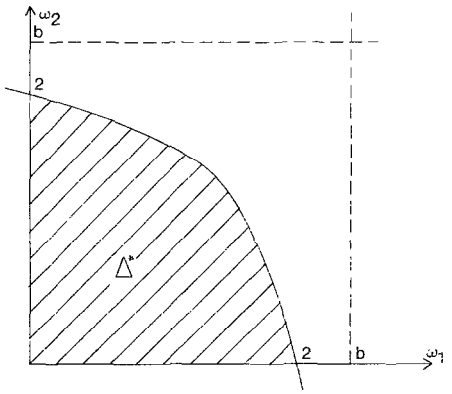

FIG. 2. Convergence domain Δ^* of the PSOR method for the biharmonic problem (I_h).

For instance, when $D=[0,1]\times[0,1]$, $\tilde{\mu}\sim\pi^{-2}h^{-2}/2$, $\tilde{\tau}\sim ch^{-1}$ [1], with c a constant depending on the boundary formula. Thus a comparison of Δ, Δ^* (Figs. 1, 2) clearly shows the favourable behavior of the convergence domain of the PSOR method; the possibilities of divergence are very limited.

(**B***) The PSOR method needs no auxiliary resolution, the coupling of (12) and (13) is satisfied for any $\Omega^{(n)}$, $\psi^{(n)}$. A SQSOR iteration needs much more computing time than a PSOR iteration particularly for the Navier–Stokes problem (II_h).

(**C**) The eigenvalue equations (20) and (22) do not allow an easy optimization of the PSOR method.

However, from the form of (I_h)—the matrix of both linear systems (12) and (13), is A—we naturally consider an optimal area to be in the neighborhood of

$$\omega_{\mathrm{opt}}(A)=\frac{2}{1+\sqrt{1-\mu_J^2}}, \tag{25}$$

where μ_J is the spectral radius of $A_J=-D^{-1}A+I$, the Jacobi matrix associated with A. We will choose ω_1 or ω_2 near ω_{opt}.

(**D**) For the Navier–Stokes problem (II_h), B is non-symmetric and does not commute with A: we cannot obtain an analytical definition of Δ^*. The convergence domain is non-symmetric and we will find an optimal region if we prescribe: $\omega_1=\omega_{\mathrm{opt}}(A)$. However, with positive schemes (centered for moderate R, upstream for large R), the behavior of the PSOR method for both problems (I_h), (II_h) will not be very different.

4. Numerical Results

Numerical experiments have been conducted for the cavity driven problem in $D=[0,1]\times[0,1]$,

$$\psi=0,\qquad \frac{\partial\psi}{\partial n}=0 \quad \text{on } \partial D \text{ except } \frac{\partial\psi}{\partial n}=-1 \text{ when } y=1. \tag{26}$$

We have considered four usual boundary formulae [3, 6, 10] which we write below on a rigid wall parallel to $\mathbf{0}y$:

(i) First order formula, Thom [10] or (1, 0) [3],

$$\Omega_{0,j} = -\frac{1}{h^2}\,[2\psi_{1,j} + \psi_{0,j+1} + \psi_{0,j-1} - 4\psi_{0,j} - 2h(\psi_x)_{0,j}]. \tag{27}$$

(ii) Second order formula (2, 0) [3],

$$\Omega_{0,j} = -\frac{1}{h^2}\,[\tfrac{1}{2}\psi_{2,j} + \psi_{0,j+1} + \psi_{0,j-1} - \tfrac{5}{2}\psi_{0,j} - h(\psi_x)_{0,j}]. \tag{28}$$

(iii) Second order formula: Jensen [10] or (2.1) [3],

$$\Omega_{0,j} = -\frac{1}{h^2}\,[-\tfrac{1}{2}\psi_{2,j} + 4\psi_{1,j} + \psi_{0,j+1} + \psi_{0,j-1} - \tfrac{11}{2}\psi_{0,j} - 3h(\psi_x)_{0,j}]. \tag{29}$$

(iv) Woods second order formula [10],

$$\Omega_{0,j} = -\frac{1}{h^2}\,[3\psi_{1,j} + \psi_{0,j+1} + \psi_{0,j-1} - 5\psi_{0,j} - 3h(\psi_x)_{0,j}] + \tfrac{1}{2}\Omega_{1,j}. \tag{30}$$

For the biharmonic problem, the signs are opposite in the above formulae. The Reynolds numbers considered in the experiments are:

$$R = 0 \quad \text{(I)} \qquad \text{and} \qquad R = 1 \text{ to } 5000 \quad \text{(II)}.$$

4.1. *Convergence Domains*

We see in Fig. 3 the convergence domains Δ (SQSOR) and Δ^* (PSOR) for the problem (I_h) with $h = 0.05$. The values of $\bar{\tau} = \rho(A^{-2}M)$ are taken in [3]

Formula	(1, 0)	(2, 0)	(2, 1)	Woods
$\bar{\tau}$	10.5	4.85	16.25	10.8

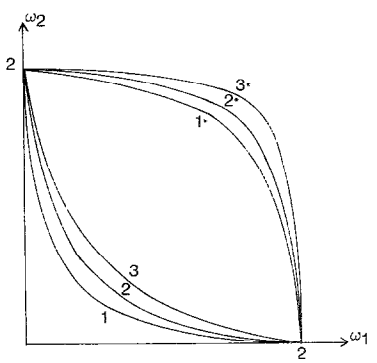

FIG. 3. Curves p, p^*, boundaries of the convergence domains Δ, Δ^* for different boundary formulae: $p = 1$, (29); $p = 2$, (27); $p = 3$, (28).

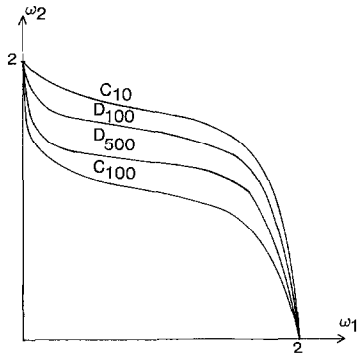

FIG. 4. Boundaries of Δ^* for different Reynolds numbers R: C_R (centered scheme), D_R (upstream scheme).

For the Navier–Stokes problem (II_h), we get an approximate value of $\rho(A^{-1}B^{-1}M)$—at the beginning of the SOR procedure—which is not very different from $\bar{\tau}$, except when R is large. In Fig. 4 we plot the convergence domains Δ^* (PSOR) given by the numerical experiments for some values of R ($h = 0.1$).

Remark. When R increases, the convergence domain decreases towards ω_2 (corresponding to B). The same behavior is observed when the PSOR method is applied to the only matrix B, for the eigenvalues of $\mathscr{L}_\omega(B)$ become complex [3, 9, 12]. Note moreover that the convergence domain is more extended when using an upstream scheme.

4.2. *Optimization of the SOR Algorithms*

Table I (centered scheme: $R \leqslant 100$) and Table II (upstream scheme: $R \geqslant 500$) give the number of outer SQSOR iterations required to solve the cavity problem (26) with one of the boundary formulae (27)–(30). Iterations were stopped when

$$\|\Omega^{(n+1)} - \Omega^{(n)}\|_\infty < 10^{-4},$$

the mesh used was $h = 0.05$ and $\omega_1 = \omega_2 = 2[1 + \sqrt{1 + \bar{\tau}}]^{-1}$.

TABLE I

Outer SQSOR Iteration Number

Boundary formulae		$\omega_1 = \omega_2$	Reynolds number					
			0	1	10	50	100	R
(27)	(1, 0)	0.46	22	22	23	32	43	N_E
(28)	(2, 0)	0.58	14	14	15	20	26	N_E
(29)	(2, 1)	0.39	27	27	29	40	56	N_E
(30)	Woods	0.45	20	20	21	34	47	N_E

Note. Iteration Number N_E when using centered scheme.

TABLE II

Outer SQSOR Iteration Number

Boundary formulae		$\omega_1 = \omega_2$	Reynolds number 500	1000	2000	5000	R
(27)	(1, 0)	0.44	71	84	118	143	N_E
(28)	(2, 0)	0.55	43	67	82	104	N_E
(29)	(2, 1)	0.37	88	109	141	185	N_E
(30)	Woods	0.44	76	90	106	132	N_E

Note. Iteration Number N_E when using upstream scheme.

For the same range of values of R, Tables III and IV give the iteration numbers of the PSOR method: we have choosen (see above Sect. 3)

$$\omega_1 = \omega_{\text{opt}}(A) = \frac{2}{1 + \sin \pi h} = 1.729$$

and numerically optimized the choice of ω_2.

These tables show that N is 2 to 4 times N_E depending on R and the boundary formulaes. One SQSOR iteration needs the solution of two large linear systems with matrices A (I_h), A and B (II_h), and in particular, for (II_h), each iteration modifies the discrete vorticity equation (matrix B).

Therefore the computing time is significantly reduced for the Navier–Stokes problem (II_h) when using the PSOR algorithm. This advantage depends on the procedure used in the inner resolution for the SQSOR method: if we use inner point SOR iterations, N_E must be multiplied by at least 10 (the number of iterations needed to obtain $\tilde{\psi}^{(n+1)}$, $\tilde{\Omega}^{(n+1)}$) and then compared with N.

Note that the usual difficulties encountered when R is large (slower convergence and restriction of the convergence domain) are not avoided by these algorithms.

TABLE III

PSOR Iteration Number

Boundary formulae		Reynolds number 0	1	10	50	100	R
(27)	(1, 0)	76(1.1)	75(1.1)	73(1.1)	75(1)	84(0.9)	$N(\omega_2)$
(28)	(2, 0)	65(1.2)	66(1.2)	68(1.15)	80(1)	87(0.9)	$N(\omega_2)$
(29)	(2, 1)	100(0.95)	98(0.95)	101(0.95)	99(0.9)	99(0.8)	$N(\omega_2)$
(30)	Woods	96(1)	96(1)	95(1)	87(0.95)	87(0.8)	$N(\omega_2)$

Note. Iteration Number N with $\omega_1 = 1.73$, ω_2 specified between brackets (centered scheme).

TABLE IV

PSOR Iteration Number

Boundary formulae		Reynolds number 500	1000	2000	5000	R
(27)	(1, 0)	123(0.9)	133(0.6)	134(0.5)	208(0.5)	$N(\omega_2)$
(28)	(2, 0)	137(0.9)	220(0.7)	245(0.7)	261(0.7)	$N(\omega_2)$
(29)	(2.1)	185(0.7)	204(0.65)	218(0.6)	227(0.6)	$N(\omega_2)$
(30)	Woods	186(0.7)	213(0.65)	227(0.6)	222(0.6)	$N(\omega_2)$

Note. Iteration Number N with $\omega_1 = 1.73$, ω_2 specified between brackets (upstream scheme).

These difficulties are more significant for the SQSOR algorithm (Figs 3, 4 [5]). We show in Fig. 5 the reliability of the PSOR method. For an example, (26) with boundary formula (27) and $h = 0.1$, we plot the "équi-itération" curves for ω_1, ω_2 close to the optimal values ($\omega_1 = \omega_{opt}(A) = 1.528$). We observe a wide optimal area, and variations of ω_1, ω_2 slightly modify the iteration number, especially when ω_2, only numerically optimized, varies, the variation of N is less than 10 % when ω_2 varies from 0.78 to 1.02 in our example.

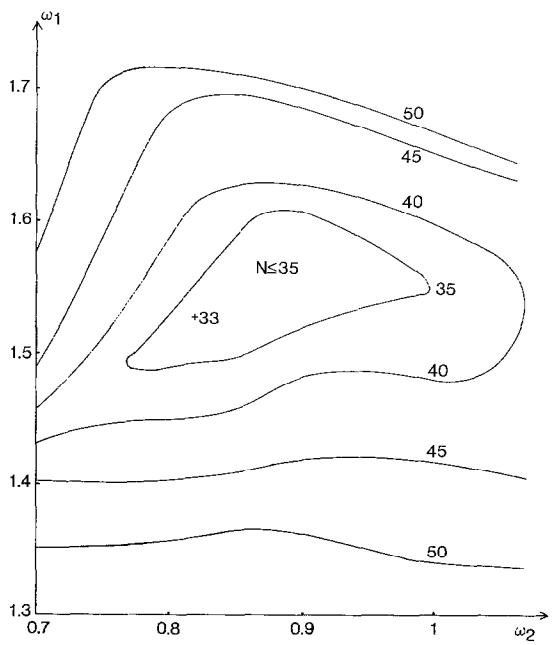

FIG. 5. "Equi-iterations" curves near the optimal values of ω_1, ω_2 for the data $R = 1$, $h = 0.1$ and the boundary formula (27).

5. CONCLUSIONS

The above studies, convergence domain, optimization and reliability of the algorithm, clearly show the efficiency of the classical point SOR method adjusted to the elliptic systems (I), (II). Numerical experiments for different values of h allowed us to observe that the rate of convergence of the PSOR method is $O(h)$ even if the choice of ω_1, ω_2 is not quite optimal. It must be remembered that the asymptotical behavior ($O(h^{1/2})$) of the SQSOR method concerns only the outer iterations, the efficiency of the method also depends on obtaining the successive approximations (direct method or inner iterations). Moreover, this best behavior is rather limited since the convergence domain of the SQSOR method becomes smaller when $h \to 0$, contrary to the convergence domain of the PSOR method. This fact notably reduces the importance of the difference between the convergence rates of both methods. For the Navier–Stokes problem (II_h), the advantage of the PSOR method may be stressed by the use of variable parameters $\omega_{i,j}$ [4, 7] depending on the values of the coefficients of the first derivatives in the vorticity equation.

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